**Physics 212: Statistical mechanics II, Fall 2012**

**Midterm**: ends 3:30 p.m., Thursday, 11/15/12

**Directions**: The allotted time is 80 minutes. The 4 problems count equally. No books or notes are allowed, and please ask for help only if a question’s meaning is unclear.

1. (5 points each) Consider a model on the square lattice with 2 Ising spins per site, $s_i = \pm 1$ and $\sigma_i = \pm 1$.

(a) Suppose that the two types of spin are decoupled from each other and have nearest-neighbor interactions with no external field:

$$E = -J \sum_{\langle ij \rangle} (s_i s_j + \sigma_i \sigma_j).$$

(1)

Use the mean-field equation for a single Ising model (you do not need to rederive it) to deduce the critical temperature of this model.

Let $m = \langle s_i \rangle$. The mean-field equation is $m = \tanh(z\beta Jm)$ with $z = 4$ on the square lattice. This has a transition at $4\beta J = 1$, or $k_B T = 4J$. The expectation value of $\sigma$ behaves identically.

(b) How many distinct ordered states are there below the critical temperature?

There are four ordered states because each type of spin can be in each of two ordered states.

(c) Now consider adding a term $-t \sum_{\langle ij \rangle} \sigma_i s_j$, where $t$ is some positive number. Derive a set of mean-field equations for $m = \langle s_i \rangle$, $n = \langle \sigma_i \rangle$.

We have

$$m = \tanh(z\beta Jm + z\beta tn), \quad n = \tanh(z\beta Jn + z\beta tm).$$

(2)

(d) How many ordered states are there at low temperature for $t > 0$?

There are now two ordered states ($m$ and $n$ both positive, and $m$ and $n$ both negative).

(e) Assume that there is a second-order transition where both $m$ and $n$ become nonzero. What is the critical temperature in mean-field theory? Is it higher or lower than in the decoupled case?

We need to solve the linear equations

$$m = z\beta Jm + z\beta tn, \quad n = z\beta Jn + z\beta tm.$$  

(3)

The solutions are the points where the matrix

$$\begin{pmatrix} z\beta J - 1 & z\beta t \\ z\beta t & z\beta J - 1 \end{pmatrix}$$

(4)

has a zero eigenvalue $\lambda$. This gives $(z\beta J - 1)^2 - z^2 \beta^2 t^2 = 0$, or $z\beta J = 1 \pm z\beta t$. This gives

$$k_B T_c = z(J \pm t).$$

(5)
Now you can do a bit of physical thinking to select the right root. The interaction between the models is unfrustrated and should enhance the ordering, so the critical temperature should be higher than in the previous model: \( k_B T_c = z(J + t) \). This makes sense at \( J = 0 \), where we again have two decoupled Ising models.

2. Short answers: 5 points each.

(a) What is the Reynolds number of a spherical object with length scale 1 cm, velocity \( 10^2 \) m/s, moving through a fluid with the density of water and viscosity \( \rho = 1 \) kilogram per meter per second? Would you expect viscous linear drag to apply?

The Reynolds number is
\[
R = \frac{\rho v L}{\mu} = \frac{(1000 \text{kg/m}^3)(10^{-2} \text{m})(10^2 \text{m/s})}{1 \text{kg/m/s}} = 10^3.
\]
Since \( R \gg 1 \), turbulent rather than linear drag will apply.

(b) For a gas of \( N \) particles with two-body interactions, define the entropy that increases in Boltzmann’s theorem about the increase of entropy.

\[
S = - \int f_1 \log f_1 \, dp \, dx,
\]
up to an arbitrary multiplicative constant.

(c) Suppose a two-dimensional system at criticality has a perturbation with RG eigenvalue \( y = 15/8 \). If this perturbation is represented in a field theory by the term
\[
\int d^2 x h \phi,
\]
what must be the scaling dimension of the field \( \phi \)? What is its correlation function \( \langle \phi(0) \phi(r) \rangle \)?

\([\phi] = 1/8, r^{-1/4}\). (These are the values for the magnetic direction of the 2D Ising model.)

(d) In which dimensions do self-avoiding walks have linear extent \( R \sim N^\nu \) with a larger exponent \( \nu \) than for random walks?

\( d = 1, 2, 3 \); in \( d = 4 \) the exponent is \( 1/2 \), as predicted by Flory theory, which is the same as the random walk value. The same is true above \( d = 4 \). (e) Does the Heisenberg model (like Ising but the spin is a 3-component unit vector) have a phase transition at nonzero temperature in two dimensions (\( d = 2 \))?

No: the Ising model has an ordered phase, but continuous order parameters have no ordered phase at \( T > 0 \) in \( d = 2 \) (Mermin-Wagner theorem); the XY model has a Kosterlitz-Thouless transition, but the Heisenberg model has no transition at all.

3. Suppose that a system is described by the rescaling map in a one-parameter space \( K' = K^{7/5} \) with \( b = 2 \). Assume \( K \geq 0 \) throughout.

(a) (4 points) What are the \( K \geq 0 \) fixed points?
(b) (8 points) At the $0 < K^* < \infty$ fixed point, linearize the rescaling transformation. What is $y_T$?

(c) (13 points) Derive the critical exponent $\nu$ that controls the divergence of the correlation length at this critical point. Hint: use the relation between $\xi'$ and $\xi$ near the critical point.

(a) $K=0, 1, \text{ and } +\infty$ are the fixed points. (b) At $K = 1$, $\frac{dR}{dK} = 7/5$, so $y_T = \log_2(7/5)$. (c) The definition of $\nu$ is that near the critical point, $\xi = A(K - K_c)^{-\nu}$. The relationship between new and rescaled lengths implies $\xi = b\xi'$, so $A(K - K_c)^{-\nu} = b A(K - K_c)^{-\nu} = b b^{-\nu y_T} A(K - K_c)^{-\nu}$, and $\nu = 1/y_T = 1/(\log_2(7/5))$.

4. (a) (10 points) Write the Boltzmann equation for a dilute gas of point particles of mass $m$ with the collision function $w$ and external force $F$.

$$ \frac{\partial f}{\partial t} + v \cdot \nabla_x f + F \cdot \nabla_p f = C(f) = \int w(f(x,p_1')f(x,p_2') - f(x,p_1)f(x,p_2)) dp_1' dp_2' dp_2. \quad (9) $$

(b) (5 points) Argue that if the particles only move in one dimension, energy and momentum conservation requires $w$ to take a simple form.

Elastic collisions in 1D can only keep the same energy and momentum or exchange them, so now $w$ has delta functions representing this. (Furthermore, since the collisions therefore don’t actually change the distribution, we can just take $w = 0$.)

(c) (5 points) Now consider a single particle moving in one dimension and let the force $F$ on the particle be random in time:

$$ \langle F(t)F(t') \rangle = K \delta(t-t'). \quad (10) $$

If there is no damping force on the particle, write an equation for its average kinetic energy as a function of time, assuming it starts with velocity $v_0$ at time $t = 0$.

Since $v(t) = v_0 + \int_0^t F(t)$,

$$ \langle v(t)^2 \rangle = v_0^2 + 2v_0 \int_0^t F(t') dt' + \left( \int_0^t \int_0^t dt' dt'' F(t')F(t'') \right) = v_0^2 + Kt \quad (11) $$

so the average kinetic energy increases linearly in time:

$$ \langle \frac{mv^2}{2} \rangle = \frac{mv_0^2}{2} + \frac{mKt}{2}. \quad (12) $$

(d) (5 points) If there is a damping force $-m\gamma v$, what is the average kinetic energy of the particle as $t \to \infty$?

The equation is solved by

$$ v(t) = v_0 e^{-\gamma t} + \frac{e^{-\gamma t}}{m} \int_0^t dt' e^{\gamma t'} F(t'). \quad (13) $$

The first term drops out as $t \to \infty$. Then similar math to in the preceding problem gives that and

$$ \langle v^2 \rangle_{t \to \infty} = \frac{K (1 - e^{-2\gamma t})}{2\gamma m^2} \to \frac{K}{2\gamma m^2}. \quad (14) $$
or that the average kinetic energy is $K/(4\gamma m)$. Units check: $K$ has units of force squared times time, or force times momentum. Dividing by mass gives force times velocity, or power, and dividing by $\gamma$ is multiplying by time, which gives energy.