Directions: The allotted time is 80 minutes. The problems count equally. No books or notes are allowed, and please ask for help only if a question’s meaning is unclear. Start your answer to each problem on a new page.

1. (a) (10 points) Consider a model like the Ising model but with three values of the spin: \( s_i = -1, 0, 1 \). Write the heuristic mean-field equation on the square lattice in two dimensions \((z = 4)\), with interaction term \( J \sigma_i \sigma_j \). Assume no magnetic field \((H = 0)\).

Considering a single spin in the field of its neighbors leads to the mean-field equation for the magnetization \( m = \langle s \rangle \)

\[
m = \frac{2 \sinh(zKm)}{2 \cosh(zKm) + 1}.
\]

Here \( K = \beta J \).

(b) (10 points) What is the critical temperature? Is this lower or higher than the critical temperature of the ordinary Ising model at the same \( J \) on the same lattice?

The critical temperature can be obtained by approximating the equation when \( m \) is small. That leads to

\[
m \approx \frac{2zKm}{3}
\]

or \( zK = 3/2 \), compared to \( zK = 1 \). So \( k_B T_c = \frac{2}{3} zJ \), and the critical temperature is reduced.

(c) (5 points) In what dimensions do you expect this model to have a phase transition? In what dimensions will it be described by mean-field theory? You do not need to justify your answers to this part.

By universality, we expect this model to be similar to the Ising model (the mean-field calculation above suggests that the effect of vacancies is not to destroy the transition but to shift its location). It will therefore have a phase transition at nonzero temperature in \( d \geq 2 \), and to be described by mean-field theory for \( d \geq 4 \).
2. Consider a dilute gas of $N \gg 1$ classical point particles in three dimensions described by a Hamiltonian with short-ranged two-particle interactions: the interaction potential is $V(|\mathbf{x}_1 - \mathbf{x}_2|)$ and the external potential is a spring giving a force $F = -kx$.

(a) (5 points) Let $f_N(0, \mathbf{x}_1, \mathbf{p}_1, \mathbf{x}_2, \mathbf{p}_2, \ldots, \mathbf{x}_N, \mathbf{p}_N)$ be the $N$-particle distribution function at $t = 0$. Suppose that at $t = 0$, $f_N = 1/V$ inside a small volume $V$ of $N$-particle phase space, and 0 elsewhere, so that the integral of $f_N$ over all phase space is 1. State Liouville’s theorem.

Liouville’s theorem is that $f_N$ is constant along particle trajectories:

$$\frac{\partial f_N}{\partial t} + \sum_{i=1}^{Nd} \left( \frac{dx_i}{dt} \frac{\partial f_N}{\partial x_i} + \frac{dp_i}{dt} \frac{\partial f_N}{\partial p_i} \right) = 0. \quad (3)$$

(b) (5 points) Write the Boltzmann equation. Define your notation clearly.

With the notation we used in class, the Boltzmann equation is

$$\partial_t f_1 + \mathbf{v} \cdot \nabla_\mathbf{x} f_1 + \mathbf{F} \cdot \nabla_\mathbf{p} f_1 = C(f) \quad (4)$$

where the collision term in the Boltzmann equation for $f(t, \mathbf{x}, \mathbf{p}_1)$ is

$$C(f) = \int w(p_1, p_2; p'_1, p'_2) \left( f(t, \mathbf{x}, p'_1)f(t, \mathbf{x}, p'_2) - f(t, \mathbf{x}, p_1)f(t, \mathbf{x}, p_2) \right) \, dp_2 \, dp'_1 \, dp'_2. \quad (5)$$

(c) (10 points) Now consider a single particle moving in one dimension, again with a spring force $F = -kx$. Assume that the particle also feels a damping force $-\gamma v$ and a random (Langevin-type) force $G(t)$. Express the power spectrum of $x$ assuming that the random force is white noise.

The equation of motion is

$$m\ddot{x} = -\gamma \dot{x} - kx + G(t) \quad (6)$$

Fourier-transforming the equation of motion leads to the following relation between the power spectrum of $x$ and of $G$:

$$I_x(\omega) = \frac{I_G(\omega)}{(m\omega^2 - k)^2 + \gamma^2\omega^2} \quad (7)$$

and white noise means that $I_G$ is constant.

(d) (5 points) If there is no spring force, the power spectrum of $x$ diverges as $\omega \to 0$ and the total power becomes ill-defined. What qualitative change in the typical trajectories does this signal? What happens when $k > 0$ and $\gamma = 0$?

The trajectories become unbounded in space if we turn off the spring–this is the standard Brownian motion problem that we solved in class, where the particle’s distribution in space becomes like that of a random walk (i.e., a Gaussian with variance linear in time). If the spring is nonzero
but there is no damping, then inspection of the power spectrum shows that total power will still diverge because of resonance if there is a nonzero $I_G(\omega)$ at the resonance frequency of the spring.

3. Short answers. No explanation is required, just a word or formula. 5 points each.

(a) The velocity autocorrelation $\langle v(t)v(t+\Delta t) \rangle$ for a Brownian particle with drag, moving under a white-noise random force, is proportional to $\delta(\Delta t)$. True or false?

False, because the drag “colors” the noise.

(b) The Heisenberg model (like Ising but the spin is a 3-component unit vector) has no ordered phase for nonzero temperature in $d = 2$. True or false?

True, because by the Mermin-Wagner theorem there is no breaking of continuous symmetry at finite temperature for $d \leq 2$.

(c) State the requirement on transition rates in Markovian dynamics that comes from the principle of detailed balance.

The probability currents between two states are equal in the two directions: $P_{i\to j}r_{i\to j} = P_{j\to i}r_{j\to i}$. Using Boltzmann distribution for the probabilities the rates satisfy the relation

$$
\frac{r_{i\to j}}{r_{j\to i}} = e^{(E_i-E_j)/k_BT}. 
$$

(d) Give an example of a configuration (of density, velocity, and pressure) that is an equilibrium of zeroth-order hydrodynamics but not physically an equilibrium.

The example we used in class was constant density and pressure, and velocity being always along $x$ but a function of $y$ (we chose arctan but any non-constant function will do): $u = u(y)\hat{x}$. Other examples are OK.

(e) How does the correlation length of the one-dimensional Ising model diverge as $T \to 0$?

Exponentially in $1/T$, since

$$
\xi = \frac{1}{-\log \tanh K} = \frac{1}{-\log(1 - 2e^{-2K})} = \frac{1}{2e^{-2K}} = \frac{e^{2K}}{2} = \frac{1}{2} e^{2J/(k_BT)}. 
$$

4. Suppose that a system with three parameters $s, t, u$ in three spatial dimensions has, close to a critical point, the linearized rescaling map (with $b = 2$)

$$
R(s, t, u) = (s', t', u'), \text{ with } \begin{pmatrix} s' \\ t' \\ u' \\ \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{5}{4} \end{pmatrix} \begin{pmatrix} s \\ t \\ u \end{pmatrix}. 
$$
(a) (7 points) How many relevant directions are there at this critical point? A picture may help but is not required.

Finding the eigenvalues of the rescaling map gives two eigenvalues greater than 1 (5/4 and 3/2) and one less than 1 (1/2). That means there are two relevant operators.

(b) (12 points) Find the singular part of the free energy density close to this critical point in terms of variables $s, t, u$. You do not need to derive your answer, but you should not include any variables that do not affect the scaling form.

The relevant operators from (a) have $y$ values

$$y_1 = \log_2(5/4), \quad y_2 = \log_2(3/2).$$

(11)

The first corresponds to the eigendirection $u$ and the second to $(s + t)/\sqrt{2}$.

The idea of the scaling form is that the singular part of the free energy does not depend on the two variables independently but on one ratio and a prefactor. We choose $u$ as the direction to use in the prefactor but the other relevant direction can be used as well. Then the scaling form is expressed compactly as

$$f_s(s, t, u) = u^{d/y_1} \Phi((s + t)/\sqrt{2})/u^{y_2/y_1}$$

(12)

where we have absorbed the constants $u_0$ etc. used in our derivation into the definition of $\Phi$. (Lecture 12 contains the derivation if you would like the intermediate steps.)

(c) (6 points) Suppose that the three eigendirections at the critical point correspond to spatial integrals of three local operators, in the same way that the magnetic direction $h$ in the Ising model corresponds to a term $h \int d^d x m$ or $h \int d^d x \phi$. For each of the three local operators, state how its correlation function $G_i(r)$ scales with $r$ at the critical point.

The relation between the scaling dimension of a local operator and its RG eigenvalue is

$$[F] = d - y_F.$$  

(13)

As an example, the field $\phi^2$ at the Gaussian critical point has $[\phi^2] = d - 2$ and RG eigenvalue $y_1 = 2$. Above we said that the system is in thee spatial dimensions, so $d = 3$. Then for the $u$ direction, call the local operator $\hat{u}$

$$[\hat{u}] = d - y_1 = 3 - \log_2(5/4).$$

(14)

The correlation function at the critical point is then

$$\langle \hat{u}(0)\hat{u}(x) \rangle \sim \left(\frac{1}{x}\right)^{2[\hat{u}]} \sim \left(\frac{1}{x}\right)^{2(3 - \log_2(5/4))}.$$  

(15)

and similarly for the other directions, with scaling dimensions $3 - \log_2(3/2)$ and $3 - \log_2(1/2) = 4$. 