Let us get some practice drawing phase diagrams of different systems in order to understand the renormalization-group ideas more generally. As an extension of the Ising model, consider a version with annealed/thermal vacancies. There are now three possible states for each site: spin up, spin down, or vacant, which we denote as $\sigma = 0$. Incidentally, if the vacancies are not thermal variables but rather static or “quenched”, then the physics is different (and more difficult). We keep the spin interaction Hamiltonian but add an energy $\Delta$ to favor or penalize vacancies:

$$H = -J \sum_{ij} \sigma_i \sigma_j + \Delta \sum_i \sigma_i^2 - H \sum_i \sigma_i.$$  \hspace{1cm} (1)

This is known as the Blume-Capel model. Here the sign of $\Delta$ is chosen so that positive $\Delta$ makes vacancies favorable. Assume $H = 0$ and consider the phase diagram. At $\Delta = -\infty$, vacancies are suppressed and we wind up with the original Ising model. At $\Delta = +\infty$, all we have are vacancies and the ground state is just vacant.

Another simple limit is zero temperature, when we need only to find the ground state of the Hamiltonian. This will contain all spins aligned, and the only question is whether the state of all spins up or down has lower energy than the state of all vacancies. With the Hamiltonian as written above, this is a competition between energy per site $\Delta - Jz/2$ (all spins up or down) and 0. At zero temperature, this is a first-order transition as a function of $\Delta/J$.

We know that the transition is second-order at $\Delta = -\infty$, and occurs at the nonzero transition temperature $T_c$ of the Ising model. This means that at some $\Delta$ there is a change between first-order transitions and second-order transitions: the point that separates the transition line is known as a tricritical point. Note that reaching a tricritical point requires “tuning” of one more experimental parameter or “knob” than an ordinary critical point.

Now we return to the Blume-Capel model and try to understand the renormalization-group flows. We said before that this model has a line of first-order transitions, a tricritical point $T_c$ and a line of second-order transitions. Where do the RG flows go in the phase diagram? At every one of the second-order transitions, the correlation length is infinite, so we cannot guess the direction of the RG flows just from the requirement that the correlation length decrease in the rescaled problem. If we are willing to assume that the ordinary 2D Ising model transition at $\Delta = -\infty$ has only the relevant directions that we already know (temperature and magnetic field), then the vacancy direction must be irrelevant. From this it follows that, unless there is a special point between the tricritical point and the 2D Ising fixed point, the flows must go from the former to the latter: the tricritical point has three relevant directions, corresponding to the three experimental parameters that must be tuned.

We next want to understand this tricritical point in the Blume-Capel model in the Landau phenomenological field theory. Suppose we are looking at the phenomenological free energy, expand in terms of the order parameter $m$: in zero magnetic field, there is a symmetry $m \leftrightarrow -m$, so

$$F(m) = \tau_0 m^2/2 + u_0 m^4 + s_0 m^6 + \ldots$$  \hspace{1cm} (2)

For an ordinary critical point, $u_0 > 0$ and the coefficient of the $m^2$ term vanishes: this is the one parameter to be tuned. A first-order transition can be obtained (this is easily seen by drawing
pictures) if \( u_0 < 0 \) instead. The special critical point of the Blume-Capel model corresponds to tuning of two parameters, which corresponds to vanishing of both the \( m^2 \) and \( m^4 \) coefficients in the Landau free energy (\( r_0 = 0 \) and \( u_0 = 0 \)): this picture correctly predicts that the tricritical point should separate first-order and second-order transitions. This Landau form has far-reaching consequences: such a “tricritical” point has different exponents in mean field theory.

This tricritical point also has upper critical dimension 3 rather than 4, so mean-field theory is exact in three dimensions, and the above scenario for the phase diagram can be confirmed in detail. As an example of a critical exponent which is different from the ordinary critical point (“bicritical”) MFT value, consider \( \delta \): now we have, adding a linear term proportional to \( H \),

\[
\frac{\partial \psi}{\partial m} = 0 \Rightarrow m \sim H^5 \Rightarrow \delta = 5
\]

when before we had \( \delta = 3 \). Similarly for \( \beta \), we now have

\[
\frac{\partial \psi}{\partial m} = 0 \Rightarrow tm \sim m^5
\]

or \( \beta = 1/4 \) instead of \( 1/2 \) in the bicritical case.

**High- and low-temperature expansions and duality of the 2D Ising model**

Why do we believe conjectures such as scaling and universality at critical points? Frequently predictions can be tested by exact results, especially on lattice models in two dimensions. We start by considering the isotropic 2D Ising model on the square lattice with rescaled coupling \( K = \beta J \):

\[
-\beta E = K \sum_{\langle ij \rangle} \sigma_i \sigma_j.
\]

At \( T = 0 \), the partition function is just \( Z = 2e^{N_b K} \), where \( N_b \) is the number of bonds of the lattice and the factor of 2 comes from spin-up and spin-down configurations. At slightly higher temperatures, we expect that the partition function will be dominated by configurations where patches of spins have flipped from the background state. We can describe such a state in terms of paths (possibly disconnected) that separate background spins from the flipped patches: if such a path \( P \) crosses \( \ell(P) \) bonds, then the energy increase relative to the ground state is \( 2J \ell(P) \). The partition function can thus be written as

\[
Z = 2e^{N_b K} \sum_P e^{-2K \ell(P)}.
\]

Note that these paths do not connect sites of the original lattice, but instead connect squares or “plaquettes”: it is natural to think of such paths as connecting sites of the “dual lattice” formed by putting a site at the center of each square of the original lattice. The square lattice is somewhat special in that its dual lattice is also a square lattice.

What about high temperatures? The partition function is

\[
Z = \sum_s e^{\sum_{\langle ij \rangle} K s_i s_j} = \sum_s \prod_{\langle ij \rangle} e^{K s_i s_j} = \sum_s \prod_{\langle ij \rangle} (\cosh K + s_i s_j \sinh K)
\]

where we have taken advantage of the fact that \( s_i s_j = \pm 1 \) in the last equality. Now rewrite this as

\[
Z = (\cosh K)^{N_b} \sum_{\langle ij \rangle} (1 + s_i s_j \tanh K)
\]
and imagine expanding the resulting polynomial in powers of \( \tanh K \). Now if a particular spin, say \( s_i \), appears an odd number of times in one term, then that term will give zero when \( s_i \) is summed over. So the only terms that survive are again closed (possibly disconnected) paths in which every site appears 0, 2, or 4 times. The partition function is then, written as a sum over paths on the original (not dual) square lattice,

\[
Z = 2^{N_s} (\cosh K)^{N_b} \sum_P (\tanh K)^{\ell(P)}.
\]  

(9)

where the factor \( 2^{N_s} \) comes from the sum over states.

Now we can compare the high-temperature and low-temperature expansions and note an interesting property. Both expansions consist of a smooth prefactor multiplying a summation over paths, and any phase transition or other singular property must result from the behavior of the summation. However, the two summations are the same if we identify

\[
e^{-2K^*} = \tanh K \Rightarrow K^* = -\frac{1}{2} \log \tanh K.
\]

(10)

where \( K^* \) is the coupling in the low-temperature expansion and \( K \) the coupling in the high-temperature coupling. There seems to be a connection (a self-duality) between the high-temperature and low-temperature behavior of this model.

Dualities like this are very important in modern physics because they connect weak coupling in one problem to strong coupling in another problem; then the perturbative methods for weak coupling in one problem can be used to get information about strong coupling in the other problem. A self-duality is a duality between different regimes of the same model. Including overall factors in the duality argument above predicts that, for the isotropic model with uniform couplings,

\[
\frac{\log Z(K)}{N} - \log \cosh(2K) = \frac{\log Z(K^*)}{N} - \log \cosh(2K^*).
\]

(11)

Here the original \( K \) and the dual \( K^* \) are related through the relation derived above, \( K^* = -\frac{1}{2} \log \tanh K \), or after some simple algebra,

\[
\sinh(2K) \sinh(2K^*) = 1.
\]

(12)

So as \( K \) increases, \( K^* \) decreases. If there is a single transition, then the thermodynamic singularities of both sides of (11) must be located at the same spot, so

\[
\sinh(2K_c) = 1.
\]

(13)

Since \( \sinh^{-1}(1) = \log(1 + \sqrt{2}) \), this predicts

\[
K_c = \frac{1}{2} \log(1 + \sqrt{2}).
\]

(14)

This result for the critical temperature was known from this Kramers-Wannier duality before Onsager’s exact solution.

More on marginal operators
Going back to the simple Ising model (no vacancies), it may be useful to explain in more
detail the meaning of relevant, irrelevant, and marginal directions/couplings/operators. (Strictly
speaking, directions in coupling-constant space are what are relevant or irrelevant, but we can think
of a direction $\hat{k}$ as describing an operator, or term in the Hamiltonian, $\lambda O_k$, where $\lambda$ is some small
number. That is,
\[ H = H_{fp} + \lambda O_k. \] (15)
Then $\lambda$ will grow under rescaling if $\hat{k}$ is a relevant direction, in which case $\lambda O_k$ is referred to as a
relevant operator.

To see that it is difficult to find a marginal operator, consider the 2D Ising model with anisotropic
couplings $J_x$ and $J_y$ for horizontal and vertical bonds respectively. It is known that anisotropy is
actually irrelevant in the RG sense for this transition, even though the transition survives with
nonzero anisotropy. For the anisotropic case, one can generalize the duality argument to show that
the transition line is described by
\[ \sinh(2K_x) \sinh(2K_y) = 1, \] (16)
where $K_x = \beta J_x$. Hence if $K_x$ is small, $K_y$ must be large at the transition. It is believed that all
of these transitions are described by the isotropic critical point at $K_x = K_y$: anisotropy in the 2D
Ising model is an example of an irrelevant perturbation.

Strictly marginal operators describe cases where both $H_{fp}$ and $H$ are fixed points, i.e., there is
a line of fixed points, and the parameter $\lambda$ moves along the line. Note that “line of fixed points” is a
stronger statement than “fixed line”: the phase boundary in the Blume-Capel model, for instance,
is preserved as a whole under the RG flow, but individual points on the line are not. To obtain a
true line of fixed points, we need a somewhat special model that has an exactly marginal operator:
by far the most important example of this rare phenomenon is the 2D XY model.