An important difference between classical and quantum mechanics is that, in quantum mechanics, the Hamiltonian determines both the dynamic and the static properties. The same Hamiltonian $H$ appears in both the density matrix $e^{-\beta H}$ and the Schrödinger equation of motion. In classical mechanics the situation is quite different. For instance, given the Ising model Hamiltonian,

$$H = -J \sum_{\langle ij \rangle} s_i s_j,$$

it is not clear how the system will evolve under time. An even harder question is to understand how such a system will approach thermal equilibrium. (For classical Hamiltonian systems the time evolution is specified, but often we have dealt with effective models like the Ising model for which the underlying Hamiltonian system is not clear.)

We would like to know, given a spin configuration (not necessarily the ground state) for the Ising model, how the configuration evolves in time. It turns out that there are many different answers to this question, and that the correct answer depends on the system to be modeled. One property that should be satisfied is that the thermal equilibrium configuration be preserved by the dynamics. This reflects a fundamental thermodynamic assumption: there is a static probability distribution, given by the Boltzmann factor, over the states of the system. Note that in spirit this is much like our work on understanding the increase of entropy in a dilute gas: we want to understand how microscopic dynamics are related to macroscopic thermodynamic principles.

The idea of a “master equation” is that the evolution of the probability distribution at time $t + dt$ should be memoryless (not depend on times prior to $t$) and determined by the probability distribution at time $t$. Let a different configurations of the spin system be denoted $\{s\}$, which stands for specific values for all the different spins. Starting at time $t$ from a configuration $\{s\}$, we will assume for the moment that over a short time interval $dt$ at most one spin of the system is flipped (that is, if spin flips occur independently, then in a very short time interval the probability of flipping flipped two spins will be much smaller than the probability of flipping one spin.). Let $w_j(\{s\})dt$ be the probability that spin $j$ is flipped in a small increment of time. Then we can write, for the evolution of the probability distribution $P(\{s\};t)$,

$$\frac{dP(\{s\},t)}{dt} = -\sum_j w_j(s_j,\ldots)P(s_j,\ldots;t) + \sum_j w_j(-s_j,\ldots)P(-s_j,\ldots;t).$$

Here the notation means that the spin configurations differ only for spin $j$, and that the rate of change in the probability of one configuration is given by the difference between transitions in and transitions out.

Above we said that the equilibrium distribution should be preserved by the dynamics. That means that $dP/dt = 0$ for the Boltzmann distribution, or

$$\sum_j \left( w_j(s_j,\ldots)e^{-H(s_j,\ldots)} - w_j(-s_j,\ldots)e^{-H(s_j,\ldots)} \right) = 0.$$
One way to guarantee that this equation is satisfied is to satisfy it for each individual spin $j$. This is known as the **principle of detailed balance**, which is essentially assuming local equilibrium for every allowed transition in order to force global equilibrium.

The principle of detailed balance can be written in terms of the local effective field $h_j = \sum_i J_{ij}s_i$ as

$$
\frac{w_j(s_j, \ldots)}{w_j(-s_j, \ldots)} = \exp(-h_js_j) / \exp(h_js_j).
$$

(4)

This works because the energy change in flipping $s_j$ is only a function of the neighboring spins: the energy is local. One assignment of rates that will satisfy this equation and also be local (the flipping rate of a spin depends only on the state of its neighbors) is

$$
w_j(\{s\}) = \frac{\Gamma}{2} (1 - s_j \tanh h_j).
$$

(5)

Here $\Gamma$ is some overall rate. This can be checked to work:

$$
\frac{w_j(s_j, \ldots)}{w_j(-s_j, \ldots)} = \frac{1 - s_j \tanh h_j}{1 + s_j \tanh h_j} = \frac{\cosh h_j - s_j \sinh h_j}{\cosh h_j + s_j \sinh h_j}.
$$

(6)

If $s_j$ is 1, this is $e^{-h_j} / e^{h_j}$; if $s_j$ is -1, then as expected we get $e^{h_j} / e^{-h_j}$.

The above choice of rates is known as **Glauber dynamics**. Note that spins flip independently and hence that the total magnetization is not conserved. Some insight into how Glauber dynamics works can be understood by considering the case of three spins with the middle spin up, and asking about the rate to flip the middle spin. The above predicts that with neighboring spins also up, the rate is $\Gamma(1 - \tanh 2\beta J)/2$. With neighboring spins down, the rate is $\Gamma(1 + \tanh 2\beta J)/2$. With one neighbor up and one neighbor down, the rate is just $\Gamma/2$.

A dynamics satisfying detailed balance can also be developed where the fundamental move is a flip of a pair of adjacent opposite spins. This dynamics, which does conserve the total magnetization, is known as **Kawasaki dynamics**. It turns out that, even though these two choices of the rates $w_j$ preserve the same Boltzmann weight, they are fundamentally different and have different universal properties.

To understand the differences between Glauber dynamics and Kawasaki dynamics, consider the Ising model in one dimension near zero temperature. There is no phase transition for $T > 0$, but the correlation length diverges and scaling properties emerge as $T \to 0$. For Glauber dynamics, we would like to know how long it takes a given spin to flip on average. Consider the three-spin system described above. If the outside spins are opposite, then the number of domain walls (boundaries between up and down regions) doesn’t change under a flip of the center spin, but the domain wall moves. If the outside spins are different, then the number of domain walls either increases by 2 or decreases by 2.

The equilibrium number of domain walls will be determined by an equilibrium between the creation and annihilation of domain walls. Start by assuming that the density of domain walls is small because the system is near zero temperature, and the correlation length is large. The density $\rho$ of walls changes according to

$$
\frac{d\rho}{dt} = -\Gamma(1 + \tanh 2\beta J)\rho^2 + \Gamma(1 - \tanh 2\beta J).
$$

(7)
Note that the factor of 2 arises because domain walls can be created either from all spins up or all spins down. Equilibrium of this equation requires
\[
\rho = \left( \frac{1 - \tanh 2\beta J}{1 + \tanh 2\beta J} \right)^{1/2} = e^{-2\beta J}.
\] (8)
This agrees with the calculation of the correlation length: at low temperature the correlation length satisfies
\[
\xi^{-1} = -\log \tanh \beta J = -\log(1 - e^{-2\beta J}) = e^{-2\beta J}.
\] (9)

The relaxation time is the average time it takes a single spin to flip. Domain walls essentially make random walks, and the amount of time for one spin to flip is of the same order as the amount of time for one domain wall to move a correlation length \(\xi\). Since the domain walls execute random walks, this amount of time goes as \(\xi^2\), so the relaxation time (essentially just a correlation length in the time direction) diverges as \(\xi \to \infty\) as a power-law in \(\xi\):
\[
\tau \sim \xi^z, \quad z = 2.
\] (10)
This defines the **dynamic critical exponent** \(z\).

What will be the value of \(z\) for Kawasaki dynamics? In Kawasaki dynamics the fundamental move is a flip of two neighboring opposite spins. The following heuristic argument for \(z\) turns out to be correct. Suppose we start with a domain of up spins of size \(\xi\). A single domain wall cannot move through the domain as in the above, but a single down spin, which is just a pair of adjacent domain walls, can random walk through the domain in a time \(\xi^2\). However, note that the relaxation time above in the Glauber model was for the whole domain to change spin, not for individual spins to briefly flip and then flip back. That means that, for a fair comparison, we need to multiply by another power of \(\xi\) to reflect the number of down spins that must be introduced. Here we are assuming that the up-spin domain is next to a down-spin domain, and that pairs of domain walls are being produced at the boundary of these two domains at a constant rate. (As discussed in class, there are short-range correlations that make the above assumptions only approximately true, but the prediction \(z = 3\) obtained below is believed to be exactly correct.)

Under these assumptions, for Kawasaki dynamics
\[
\tau \sim \xi^z, \quad z = 3.
\] (11)

The normal way to express this is that Glauber and Kawasaki dynamics have the same static critical exponents but are in different dynamic universality classes with different values of \(z\). At quantum critical points, \(z\) is a fundamental property that cannot be separated so easily from the other critical exponents.

An understanding of how quantum dynamics can be very interesting and important is the simple case of the quantum Heisenberg ferromagnet or antiferromagnet:
\[
H = -J \sum_{\langle ij \rangle} s_i \cdot s_j
\] (12)
where now these are quantum spin-1/2 operators. First note that quantum mechanics breaks the symmetry between ferromagnet and antiferromagnet that exists in the classical case on a
bipartite lattice. (In the classical case, one can just flip the spins on one sublattice to map the ferromagnet into the antiferromagnet; this shows that they have the same critical temperature, critical exponents, etc.). For example, two classical Ising spins with an interaction $J$ form two doublets related by symmetry, while two Heisenberg spins form a singlet and triplet that can’t be related by any simple symmetry, since one has one state and one has three states.

A profound consequence of this difference is that spin waves in a ferromagnet have the dispersion relation

$$\omega \sim k^2, \quad (13)$$

while those in an antiferromagnet have

$$\omega \sim |k|. \quad (14)$$

A handwaving way (cf. Anderson, Basic Notions of Condensed Matter Physics) to understand the difference is by thinking of the ferromagnet as a state that breaks time-reversal symmetry, while on the long length scales of a spin wave, the antiferromagnet doesn’t strongly break time-reversal symmetry (the mean magnetization is zero). This isn’t utterly convincing, but the full calculation for the antiferromagnetic case is beyond the scope of this course.