Physics 212: Statistical mechanics II  

Lecture XVII  

The last lecture had a number of new ideas, so let’s quickly review the main ones. The definition of the “scaling dimension” of a local operator or field like \( \phi(x) \) at a critical point was through its correlation function:

\[
[\phi] = \Delta \text{ if } \langle \phi(x)\phi(0) \rangle = \frac{1}{x^{2\Delta}}.  
\]  

(1)

For the Gaussian model, scaling dimensions are the same as engineering dimensions. In general, the engineering dimension of a field is minus the length dimension it would have to have to make the “action” (the quantity appearing in the exponent of \( Z = \int e^{-S} \)) have zero length dimension.

So for the Gaussian model, \( [\phi] = (d - 2)/2 \) in any dimension. Let us now give a derivation of the rescaling equations in the Gaussian model that we conjectured last time,

\[
t' = b^2 t \\
u' = b^{4-d} u \\
v' = b^{6-2d} v \\
h' = b^{d/2+1} h,  
\]  

(2)

by imitating the steps that we followed to carry out rescaling transformations on the lattice. Our starting point is the partition function

\[
Z = \int (D\phi) e^{-\int d^dx \left[ \frac{1}{2}(\nabla \phi)^2 + a^{-2} t \phi^2 + a^{-4} u \phi^4 + a^{-d/2-1} h \phi \right]}  
\]  

(3)

We want to construct a transformation that will leave the Gaussian model invariant if \( t = u = v = h = 0 \), as that is the fixed point we are attempting to describe. **First step:** sum over some variables in order to generate an effective action for the remaining variables. The idea of the momentum-shell renormalization group is: starting from a theory with an ultraviolet (short-distance) cutoff \( a^{-1} \), integrate out the momentum components with \( (ba)^{-1} \leq |k| < a^{-1} \), for some rescaling parameter \( b > 1 \).

Writing the integration measure in Fourier space, we have

\[
Z = \int |k|<a^{-1} d\phi(k) e^{-\int \psi d^dx} = \int |k|<(ba)^{-1} d\phi(k) \left( \int (ba)^{-1} < |k| < a^{-1} d\phi(k) e^{-\int \psi d^dx} \right)  
\]  

(4)

If \( \psi \) has only the two quadratic terms in it, then it has a very nice property: its integral is diagonal in momentum components, so that the integration over higher momentum components does not create any new nontrivial dependences on lower momentum components. (This separation of momentum scales is, we shall see, specific to the Gaussian model.) Assume that we are close to the Gaussian critical point and carry out the integrals over higher momentum components, which gives some overall factor.

The part of the action containing nonintegrated momentum components is the same as before, except that the momentum components do not range quite as high as before. **Second step:** The new problem is not directly comparable to the original problem because the cutoff is different. We
would like to have a rescaling map that lives in the space of problems with the same cutoff. We can achieve this by rescaling length units so that the momentum cutoff is restored to $a^{-1}$: looking at the Gaussian part in real space after the integration, we have

$$\int \psi \, d^d x = \int \left( (\nabla \phi)^2 / 2 + a^{-2} t \phi^2 \right) \, d^d x.$$  

(5)

We want to rescale length and the field $\phi$ so that the spatial cutoff is restored to $a$ from $ba$ and the critical part of the action is form-invariant. The new spatial variable should be $x' = x/b$, and to keep the first term’s coefficient the same as before we must also introduce the field variable $\phi'$, with $\phi = \phi' b^{-(d-2)/2}$. (This is another meaning of our statement before about the scaling dimension of the field $\phi$.) Now we have

$$\int \left( (\nabla \phi')^2 / 2 + a^{-2} b^2 t (\phi')^2 \right) \, d^d x'.$$

(6)

This takes the same form as the original action, but with a renormalized $t' = b^2 t$, as claimed above. The same logic applies for the other terms in the action.

In order to understand an example where scaling dimensions are not equal to engineering dimensions, because of the existence of “anomalous dimensions”, consider the Ising model in 2D. We will not be able to solve this theory here, but will interpret the values of critical exponents known from Onsager’s solution in terms of scaling dimensions $[\phi]$ and $[\phi^2]$, in order to see the difference from the Gaussian model.

Below four dimensions, the quartic term is relevant, so we need to include it in the critical theory:

$$Z = \int (D\phi) e^{-\int d^d x \left[ \frac{1}{2} (\nabla \phi)^2 + a^{-2} t \phi^2 + a^{d-4} u \phi^4 + a^{-d/2-1} h \phi \right]}$$

(7)

At the critical point of the 2D Ising model, our previous values of the critical exponents followed from $y_t = 1$ and $y_h = 15/8$. These imply

$$[\phi] = \frac{1}{8},$$

$$[\phi^2] = 1.$$  

(8)

These follow from the useful relation between the scaling dimension of a field (before being integrated over space) and its RG eigenvalue,

$$[F] = d - y_F.$$  

(9)

Do you understand where this comes from? We want to argue that $y_F = d - [F]$, which is the negative of the scaling dimension of $\int d^d x \, F$. Following the same steps as above for the Gaussian model, we see that positive $y_F$ corresponds to a relevant operator (negative scaling dimension of $\int d^d x \, F$), and that $y_F$ is the RG scaling eigenvalue associated with the field (recall that $y_t$ corresponds to the quadratic $\phi^2$ term, and $y_h$ corresponds to the $\phi$ term), $d$ is the spatial dimensionality, and $[F]$ is the scaling dimension of the field. The coefficient of a field behaves under rescaling in our old language like

$$\lambda' = b^{y_F} \lambda,$$

(10)

and in our new language like

$$\lambda' = b^{d-[F]} \lambda.$$  

(11)
Then the values above for \( \phi \) and \( \phi^2 \) occur since for 2D Ising \( y_t = 1 \) and \( y_h = 15/8 \).

Now to say that the scaling dimension \( \phi \) is equal to 1/8 implies that the correlations of \( \phi \) must fall off at criticality as
\[
\langle \phi(r)\phi(0) \rangle \sim r^{-2\Delta} = r^{-1/4}. \tag{12}
\]
This corresponds to \( \eta = 1/4 \) from our previous definition of critical exponents. However, the symbol \( \sim \) above must include some proportionality constant with nonzero length dimensions, since otherwise the two sides have different units! The left side has 0 powers of length since the field \( \phi \) has engineering dimension 0 in two dimensions, while the right side has \(-1/4\) powers of length. To get the units right, we should write
\[
\langle \phi(r)\phi(0) \rangle \sim r^{-2\Delta} = a^{1/4}r^{-1/4}. \tag{13}
\]
Here \( a \) is some length scale: at criticality the only length scale is the short-distance cutoff, so this must set the scale for \( a \). For a lattice problem we can take \( a \) to be the lattice spacing. We can give an interpretation of RG flows, common in particle physics, as rescaling of the short-distance cutoff \( a \).

The above is an example of anomalous dimensions: because the scaling dimension does not agree with the engineering dimension, additional powers of the short-distance cutoff must appear in a correlation function. This explains why, in the scaling form for the correlation function introduced several lectures ago, it was necessary in general to allow a dependence on both \( r/\xi \) and \( r/a \). The physical meaning of anomalous dimensions is quite deep: it means that near criticality the interesting physics does not take place only on the long length scale \( \xi \), but on all the scales between \( a \) and \( \xi \). All these scales are coupled in an interacting theory.

These dimensions for the 2D Ising model can’t be obtained simply from power counting. The reason why scaling dimensions were so simple for the Gaussian model is that all the integrals needed to evaluate any correlation function in the Gaussian model, even a complicated one like
\[
\langle (\phi(x)^2 - \langle \phi^2 \rangle)(\phi(0)^2 - \langle \phi^2 \rangle) \rangle, \tag{14}
\]
could be done using Wick’s theorem. In an interacting theory, calculating correlation functions like the above is very difficult.

Let’s formalize our understanding of the Gaussian model in a few simple statements. 1. Above \( d = 4 \) the Gaussian fixed point is stable under addition of a \( \phi^4 \) term. 2. Between \( d = 3 \) and \( d = 4 \), the Gaussian fixed point is unstable to a \( \phi^4 \) term, but stable to a \( \phi^6 \) term. 3. The Gaussian model has \( y_t = 2 \) and \( y_h = d/2 - 1 \).

Recall that mean-field theory has \( y_t = 2 \) and \( y_h = 3 \). Therefore the exponents of the Gaussian model agree with those of mean-field theory in \( d = 4 \). Note that they don’t agree above \( d = 4 \), when the mean-field exponents are correct but the Gaussian exponents are not; for these dimensions \( \phi^4 \) is an example of a “dangerous irrelevant operator”, as discussed in Cardy and other books on RG methods. The language means that, even though \( \phi^4 \) is irrelevant at the critical point and does not affect those critical exponents defined at criticality, it still affects others because the nature of the ordered phase is wrong if \( \phi^4 \) is not included. That is, the Gaussian model does not make sense if \( \phi^4 \) is neglected in the ordered phase where the coefficient of \( \phi^2 \) is negative, since then the free energy is not bounded below.
We would like to find a relatively simple example of how anomalous dimensions emerge in interacting models like the Ising model in 2 and 3 dimensions. The difficulty is in finding a **controlled expansion in a small parameter** so that, e.g., the critical point is only slightly different from the Gaussian model, that we now understand.

**Interacting field theories: the language of modern physics**

Now we will briefly explain how anomalous dimensions emerge in a controlled expansion of an interacting field theory. **Remark:** Note that many classical field equations, such as Maxwell’s equations, describe configurations of field theories that minimize some action: the effect of quantum mechanics is to generate fluctuations around the classical field configuration. In the same way, the theories we are discussing have thermal fluctuations of the observables, and the language of statistical field theories has considerable overlap with that of quantum field theories (we will have a bit more on this to say later). The last thirty or forty years have seen a high degree of interaction between developments in high-energy physics and in statistical physics, because of the parallels between quantum fluctuations and thermal fluctuations.

The remainder of this lecture summarizes an elegant but somewhat technical approach to the $\phi^4$ theory, which is much more complicated than the Gaussian model we’ve dealt with before. The definition of the “scaling dimension” of a local operator or field like $\phi(x)$ at a critical point was through its correlation function:

$$[\phi] = \Delta \text{ if } \langle \phi(x)\phi(0) \rangle = \frac{1}{x^{2\Delta}}.$$  \hspace{1cm} (15)

For the Gaussian model, scaling dimensions are the same as engineering dimensions. In general, the engineering dimension of a field is minus the length dimension it would have to have to make the Hamiltonian dimensionless.

So for the Gaussian model, $[\phi] = (d - 2)/2$ in any dimension and power-counting works. We would like to find a relatively simple example of how anomalous dimensions emerge in interacting models like the Ising model in 2 and 3 dimensions. The difficulty is in finding a **controlled expansion in a small parameter** so that, e.g., the critical point is only slightly different from the Gaussian model, that we now understand.

The most famous example of a controlled expansion around the Gaussian model is known as the Wilson-Fisher approach. We know that below four dimensions the $\phi^4$ term is relevant and needs to be included. However, just below four dimensions (say, in $4 - \epsilon$ dimensions), we might expect that the new “Wilson-Fisher” fixed point to which the Gaussian fixed point is unstable along the $\phi^4$ diagram (see picture) is only slightly different from the Gaussian fixed point.

Here the small parameter is $\epsilon$, which measures how far below four dimensions the system is. One of our best theoretical methods for obtaining exponents for the Ising model is by obtaining a power series for $\epsilon$, then setting $\epsilon = 1$ so $d = 3$. It is even possible, using some mathematical tricks to rewrite the power series, to set $\epsilon = 2$ and estimate the critical exponents in $d = 2$, and compare them to the exact results for the Ising model.

To start, recall the rescaling equation for different coefficients at the Gaussian fixed point:

$$t' = b^2 t$$
$$u' = b^{4-d} u$$
$$v' = b^{6-2d} v$$
Suppose that we carry out an infinitesimal rescaling: \( b = 1 + d \ell \). Then, for example,

\[
dt = 2 d\ell t \Rightarrow \frac{dt}{d\ell} = 2t,
\]

and so on for the other equations. Note that when we write an infinitesimal RG equation, the quantity \( y_t = 2 \) that was in the exponent before is now just a coefficient. Now we can ask what the RG equations should look like just below four dimensions, in order for us to obtain a new Wilson-Fisher fixed point.

There are two techniques commonly used to find the RG equations for small \( \epsilon \) (this is known as a perturbative RG, since it is perturbative in \( \epsilon \)). One can calculate correlation functions perturbatively in \( \phi^4 \) theory (i.e., when the coefficient of the \( \phi^4 \) term is small, as expected since we are just below the critical point) using the technology of Feynman diagrams. Another way, which requires fewer integrals to get the order-\( \epsilon \) result but whose physical meaning is a little less transparent, is the “operator product expansion” discussed in Cardy.

Here we will give an overview of the Feynman diagram calculation done in Goldenfeld. Perturbative calculations are the main subject of an introductory course in quantum field theory, and form just the first step of the Wilson-Fisher expansion, so we will just give an overview. In order to treat the interaction term \( \phi^4 \), we can do perturbation theory in its coefficient \( u \). In Fourier space, the \( \phi^4 \) term contains four (possibly different) momenta, but they are forced to sum to zero by the integration over real space: it looks like

\[
u \sum_{k_1, k_2, k_3} \phi(k_1)\phi(k_2)\phi(k_3)\phi(-k_1 - k_2 - k_3)
\]

Feynman diagrams are representations of terms in perturbation theory around a Gaussian fixed point. Lines represent “propagators” \( \langle \phi(k)\phi(-k) \rangle \) evaluated from the Gaussian theory, while vertices represent insertions of the perturbation terms. The vertex corresponding to the \( u \) term above is a point with four lines coming out of it, and calculating a Feynman diagram comes down to evaluating integrals over all the momenta on internal propagator lines.

The magic of the interacting theory is that different momentum scales mix in the following way: there are diagrams in the perturbation-theory expansion for a term like \( u \) that contain both momenta above the new cutoff and below the new cutoff. As a consequence, coefficients in the action with the new cutoff receive modifications from the energy scales that are integrated out, and these modifications can be kept track of using Feynman diagrams. There are also modifications because \( t \), the mass term, enters into the propagator.

Now we just focus on the resulting equations. We restrict for now to zero magnetic field (\( h = 0 \)). There are two features of the interacting theory that aren’t present in the Gaussian theory. First of all, there are terms proportional to \( t^2, u^2, tu \) in the new flow equation for \( u \), in addition to the \( du/d\ell = \epsilon u \) from the Gaussian model. The result of a moderately serious calculation to next order beyond power-counting is (cf. Cardy or Goldenfeld)

\[
\frac{du}{d\ell} = \epsilon u - t^2 - 16tu - 72u^2 + \ldots
\]
Only some of these terms will be needed in what follows. Note that the linear term here we could get simply from our knowledge about the Gaussian model, but the quadratic terms require a more serious perturbative calculation. Similarly, the linear term of the $t$ equation follows from the Gaussian model, but the quadratic terms are more complicated:

$$\frac{dt}{d\ell} = 2t - 4t^2 - 24tu - 96u^2 + \ldots.$$  \hspace{1cm} (20)

Clearly one fixed point of the two equations is $t = u = 0$, the Gaussian critical point. Now let’s look for a solution to order $\epsilon$ for another fixed point, meaning that the errors are $\epsilon^2$ or smaller. A quick inspection shows that the other solution to this order has $u^* = \epsilon/72 + O(\epsilon^2)$ and $t = 0 + O(\epsilon^2)$.

In order to get the exponents at the new fixed point, we have to linearize the RG: and critical exponents to order $\epsilon$. The $t$ equation when linearized will give the thermal eigenvalue $y_t$:

$$\frac{dt}{d\ell} \Big|_{t=0,u=\epsilon/72} = (2 - \epsilon/3)t.$$  \hspace{1cm} (21)

Hence instead of $y_t = 2$, for MFT or the Gaussian model in $d = 4$, we now have $y_t = 2 - \epsilon/3$, or

$$\nu = \frac{1}{2 - \epsilon/3} \approx \frac{1}{2} + \frac{\epsilon}{12}.$$  \hspace{1cm} (22)

The other critical exponents, found using the above equations and also the $h$ equation (actually $y_h = 3 - \epsilon/2 + O(\epsilon^2)$), at this order are $\eta = 0$,

$$\alpha = 2 - d\nu = 2 - (4 - \epsilon)(\frac{1}{2} + \frac{\epsilon}{12}) = \epsilon/6,$$
$$\beta = (d - y_h)\nu = (1 - \epsilon/2)(\frac{1}{2} + \frac{\epsilon}{12}) = \frac{1}{2} - \frac{\epsilon}{6},$$
$$\gamma = (2y_t - d)\nu = (2)(\frac{1}{2} + \frac{\epsilon}{12}) = 1 + \frac{\epsilon}{6},$$
$$\delta = \frac{3}{(1 - \epsilon)} = 3 + 3\epsilon.$$  \hspace{1cm} (23)

These can be compared to results in 3D from numerical results, such as high-temperature expansions. The $\epsilon-$expansion has been carried out to quite high order for standard models such as $\phi^4$ theory. Another common expansion is “large-$N$”: one expands in the symmetry of the problem, as the Gaussian model turns out to be valid even in three dimensions for the $O(N)$-symmetric model in the limit of large $N$. 

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