In this lecture we introduce the theory of dynamical critical phenomena, which generalizes the previous examples of Glauber and Kawasaki dynamics on the \( d = 1 \) Ising model. We concluded from that simple case that one static universality class can have multiple dynamical universality classes, and that conservation laws play a key role in long-time dynamics.

We would like to understand the problem of near-equilibrium dynamics near critical points (“dynamical critical phenomena”) in a more general way. This challenge will involve combining the Langevin dynamics introduced in the theory of Brownian motion with many ideas from static critical phenomena, including the Landau free energy and the notion of a scaling form. (This lecture will use the notation of the classic Rev. Mod. Phys. article of Hohenberg and Halperin; I will attempt to point out differences from our previous notation.) Start with a theory of a single scalar field \( \phi(x,t) \), representing for example the coarse-grained magnetization density in an Ising model. If \( H \) is a static energy density function on \( \phi \), consider the dissipative dynamical equation

\[
\frac{\partial \phi}{\partial t} = -\Gamma \frac{\partial H}{\partial \phi} + \zeta(x,t).
\]

(1)

Here the first term on the right side describes overdamped relaxation to the minimum of energy. The rate \( \Gamma \) is assumed to be real and positive. The term \( \zeta(x,t) \) is a Langevin random force, with correlations we assume to be described by

\[
\langle \zeta(x,t)\zeta(x',t') \rangle = 2\Gamma A \delta(x-x')\delta(t-t').
\]

(2)

The average here is an average over realizations of the random force. This model for critical dynamics of a single nonconserved field is referred to as “Model A”. As in our discussion of the Langevin theory of Brownian motion, we would like this average over random forcing to reproduce thermal equilibrium; actually this fixes the constant \( A \) to be just the temperature, no matter whether the energy function \( H \) is simple or complicated.

**Not covered in class:** This surprising result will be derived once we find a slightly more complicated equation, the Fokker-Planck equation, describing the evolution of a probability density \( P(p,t) \). (In the Brownian motion case treated earlier in the course, we took advantage of linearity of \( H \) to derive the connection between \( \zeta \) and forcing quickly.) Let us go back to a single particle and try to understand how temperature appears even if the equation is nonlinear. The equation of motion is assumed overdamped as in the formula above, so the momentum evolves as

\[
\frac{\partial p}{\partial t} = -\Gamma \frac{\partial H}{\partial p} + \zeta(t).
\]

(3)

Let the conditional probability to find a particle at momentum \( p \) and time \( t \), given that it started at \( p_0 \) at time \( t_0 \), be written

\[
P(p,t|p_0,t_0) = \langle \delta(p - p(t)) \rangle_{p_0,t_0}.
\]

(4)
An important property of these conditional probabilities is that we can break up any interval of
time into two subintervals, and integrate over all momenta at the intermediate time that separates
the two subintervals:

\[ P(p, t + \Delta t|p_0, t_0) = \int dp' P(p, t + \Delta t|p', t)P(p', t|p_0, t_0). \]  

(5)

The equation of motion can be integrated over a short time \( \Delta t \):

\[ p(t + \Delta t) = p' - \Gamma \frac{\partial H}{\partial p'} \Delta t + \int_t^{t+\Delta t} dt' \zeta(t'). \]  

(6)

We will use this to rewrite \( P(p, t + \Delta t|p', t) \). Now

\[ P(p, t + \Delta t|p', t) = \langle \delta(p - p(t + \Delta t)) \rangle_{p', t} = \langle \delta(p - p' + \Gamma \frac{\partial H}{\partial p'} \Delta t - \int_t^{t+\Delta t} dt' \zeta(t')) \rangle. \]  

(7)

We can expand the \( \delta \) function in a Taylor series around \( p - p' \) to quadratic order, which gives

\[ \delta(p - p' + x) = \delta(p - p') + x \frac{\partial}{\partial p} \delta(p - p') + \frac{x^2}{2} \frac{\partial^2}{\partial p^2} \delta(p - p') + \ldots. \]  

(8)

We would like to keep all terms up to order \( \Delta t \) that will survive the averaging. The term involving
the integral over random forcing averages to zero, but its square does not average to zero, since

\[ \left\langle \left( \int_t^{t+\Delta t} dt' \zeta(t') \right)^2 \right\rangle = \int_t^{t+\Delta t} dt_1 dt_2 \langle \zeta(t_1)\zeta(t_2) \rangle = 2\Gamma \Delta t. \]  

(9)

This shows that its square is in fact of order \( \Delta t \), so we need to keep it. Thus

\[ P(p, t + \Delta t|p', t) = ((1 + \Gamma \frac{\partial H}{\partial p'} \Delta t \frac{\partial}{\partial p} + A\Gamma \Delta t \frac{\partial^2}{\partial p^2})\delta(p - p')). \]  

(10)

Substitution of this into the integral over intermediate states above gives the Fokker-Planck equation,

\[ \frac{dP}{dt} = \Gamma \frac{\partial}{\partial p}(1 + \frac{\partial H}{\partial p'} \Delta t \frac{\partial}{\partial p} + A\Gamma \Delta t \frac{\partial^2}{\partial p^2})P. \]  

(11)

(One has to be a bit careful about where the derivatives act in getting to this step.) Now this will
vanish if

\[ \frac{\partial P}{\partial p} = - \frac{\partial H}{\partial p} P/A, \]  

(12)

so

\[ \frac{\partial \log P}{\partial p} = (1/A) \frac{\partial H}{\partial p} \Rightarrow P \propto \exp(-H/A). \]  

(13)

This is the desired formula: the thermal distribution will be an equilibrium as long as \( A \) is chosen
to be \( T \) (here \( k_B = 1 \)), even if \( H \) is nonlinear. The same argument goes through if we allow spatial
dependence in the problem. **End remark**

Let us now go back to Model A. For comparison, we can also discuss a model for a single
conserved field. We know that the diffusion equation conserves total particle number, so consider
the following equation, which reduces to the diffusion equation if \( H \) has only a \( r \phi^2 \) term:

\[ \frac{\partial \phi}{\partial t} = \lambda \nabla^2 \phi \frac{\partial H}{\partial \phi} + \zeta. \]  

(14)
The first term on the right side is a total divergence, so $\int \phi$ is conserved. If we note that model A is obtained from model B by replacing $\Gamma$ by $\lambda \nabla^2$, it makes sense that the new condition on $\zeta$ is

$$\langle \zeta(x,t)\zeta(x',t') \rangle = -2T \lambda \nabla^2 (\delta(x-x')\delta(t-t')).$$ (15)

The appearance of the $\lambda \nabla^2$ here means that $\zeta$ is anticorrelated in space in such a way that the average squared change induced in $\phi$ in a region is a total derivative, consistent with $\phi$ conservation.

In these models, we would like to compute the frequency- and wave-dependent susceptibility $\chi(q,\omega,r)$, where $r$ is the dimensionless distance from the critical point in the thermal direction, which we previously called $t$. This quantity gives the response in the order parameter (e.g., magnetization) to an applied field at the specified wavevector and frequency (the assumptions of linear response guarantee that the response is at the same wavevector and frequency). The expected scaling form for this quantity is

$$\chi(q,\omega,r) = b^{2-\eta} \chi(bq, b^z \omega, b^{1/\nu} r).$$ (16)

To check this, suppose that $q = \omega = 0$; then the statement is that

$$\chi(r) = b^{2-\eta} \chi(b^{1/\nu} r) \Rightarrow \chi \sim r^{-(2-\eta)\nu} \Rightarrow \gamma = (2 - \eta)\nu,$$ (17)

one of the static scaling relations we derived earlier. If we compute the response for model A to the Langevin force, we get, for the Gaussian model with free energy density $r\phi^2/2 + c(\nabla \phi)^2/2$

$$-i\omega \phi(q,\omega) = -\Gamma(r + cq^2) \phi(q,\omega) + \zeta(q,\omega) \Rightarrow \phi(q,\omega) = \frac{\zeta(q,\omega)}{-i\omega + \Gamma(r + cq^2)}.$$ (18)

The susceptibility to an applied magnetic field $h(q,\omega)$ is defined to include a constant factor $\Gamma$, as the term added to the Langevin equation is $h\Gamma$, so

$$\chi(q,\omega) = \frac{1}{-i\omega + (r + cq^2)} = \frac{1}{-i\omega + \chi^{-1}(q)}.$$ (19)

In model B, we replace $\Gamma$ by $\lambda q^2$, giving

$$\chi(q,\omega) = \frac{1}{-i\omega + (r + cq^2)} = \frac{1}{-i\omega + \chi^{-1}(q)}.$$ (20)

We allowed the static susceptibility to include the anomalous dimension $\eta$ in its scaling form, hence going beyond mean-field theory for the statics. We will now compute the dynamical critical exponent $z$ under the assumption that there is no anomalous dimension for the rates $\Gamma$ and $\lambda$, i.e., that they can be held constant as the critical point is approached. (This does not hold in general; just as anomalous dimensions emerge at critical points in statics, they can also emerge in the dynamical exponent $z$.) For model A, we have

$$\chi^{-1}(bq, b^z \omega, b^{1/\nu} r) = \frac{-ib^z \omega}{\Gamma} + \chi^{-1}(bq, b^{1/\nu} r) = \frac{-ib^z \omega}{\Gamma} + b^{2-\eta} \chi^{-1}(q, r).$$ (21)

This satisfies the desired scaling form if and only if $z = 2 - \eta$. 3
If we carry out the same calculation for model B, we obtain $z - 2 = 2 - \eta$ or $z = 4 - \eta$, and clearly the conservation law has made a difference. These “van Hove” results should be regarded in the same spirit as mean-field theory. An additional complication in dynamics, beyond the requirement of correctly describing conservation laws, is the requirement that one should correctly describe “Poisson brackets” in the dynamics. Unlike the purely dissipative models described here, models with nonzero Poisson brackets, as for example in the isotropic ferromagnet, can support propagating modes rather than only overdamped modes.